

The trace norm of r -partite graphs and matrices

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Abstract

The trace norm $\|G\|_*$ of a graph G is the sum of its singular values, i.e., the absolute values of its eigenvalues. The norm $\|G\|_*$ has been intensively studied under the name of *graph energy*, a concept introduced by Gutman in 1978.

This note studies the maximum trace norm of r -partite graphs, which raises some unusual problems for $r > 2$. It is shown that, if G is an r -partite graph of order n , then

$$\|G\|_* < \frac{n^{3/2}}{2} \sqrt{1 - 1/r} + (1 - 1/r) n.$$

For some special r this bound is asymptotically tight: e.g., if r is the order of a real symmetric conference matrix, then, for infinitely many n , there is a graph G of order n with

$$\|G\|_* > \frac{n^{3/2}}{2} \sqrt{1 - 1/r} - (1 - 1/r) n.$$

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1 Introduction

The *trace norm* $\|A\|_*$ of a matrix A is the sum of the singular values of A ; it is also known as the *nuclear norm* or the *Schatten 1-norm* of A . The trace norm of the adjacency matrix of graphs has been much studied under the name *graph energy*, a concept introduced by Gutman in [4]; for an overview of this vast research, see [5]. Thus, let us write $\|G\|_*$ for the trace norm of the adjacency matrix of a graph G , and note that $\|G\|_*$ is just the sum of the absolute values of the G eigenvalues.

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Koolen and Moulton [9] studied the maximum trace norm of graphs of order n ; in particular, they proved that if G is a graph of order n , then

$$\|G\|_* \leq n^{3/2}/2 + n/2, \quad (1)$$

with equality if and only if G belongs to a certain family of strongly regular graphs; in [6] Haemers showed that these graphs arise from a class of Hadamard matrices. Furthermore, Koolen and Moulton [10] proved that if G is a bipartite graph of order n , then

$$\|G\|_* \leq n^{3/2}/\sqrt{8} + n/2, \quad (2)$$

with equality if and only if G is the incidence graph of a particular type of design.

Given the cases of equality in bounds (1) and (2), arguably, much of their thrill is in the fact that the bulk parameter “trace norm” is maximized on rare graphs of delicate structure.

To make the next step in this direction, recall that a graph is called r -partite if its vertices can be partitioned into r edgeless sets. We shall study the following natural problem arising in the vein of (2):

Problem 1 *If $r \geq 3$, what is the maximum trace norm of an r -partite graph of order n ?*

For complete r -partite graphs the question was answered in [11], but in general Problem 1 is much more difficult than the question for bipartite graphs, for it has many variations, it requires novel constructions, and most of it is beyond the reach of present methods.

First, we shall restate Problem 1 in analytic matrix form and shall give some upper bounds. The matrix setup elucidates the main factors in the graph problem. Further, using graph-theoretic proofs, we shall fine-tune these upper bounds at the price of somewhat increased complexity.

We shall show that for infinitely many r our bounds are exact or tight up to low order terms. The intriguing point here is that the tightness of the bounds is known only if r is the order of a conference matrix, and since such matrices do not exist for all r , a lot of open problems arise.

2 Upper bounds

Given an $n \times n$ matrix $A = [a_{i,j}]$ and nonempty sets $I \subset [n]$ and $J \subset [n]$, write $A[I, J]$ for the submatrix of all $a_{i,j}$ with $i \in I$ and $j \in J$. An $n \times n$ matrix A is called k -partite if there is a partition of its index set $[n] = N_1 \cup \dots \cup N_k$ such that $A[N_i, N_i] = 0$ for any $i \in [k]$.

Further, write A^* for the *Hermitian transpose* of A , and let $\|A\|_{\max} = \max_{i,j} |a_{i,j}|$. As usual, I_n and J_n stand for the identity and the all-ones matrices of order n ; we let $K_n = J_n - I_n$.

Theorem 2 *Let $n \geq r \geq 2$, and let A be an $n \times n$ complex matrix with $\|A\|_{\max} \leq 1$. If A is r -partite, then*

$$\|A\|_* \leq n^{3/2} \sqrt{1 - 1/r}. \quad (3)$$

Equality holds if and only if all singular values of A are equal to $\sqrt{(1 - 1/r)n}$.

Proof Let $A = [a_{i,j}]$, and let $\sigma_1, \dots, \sigma_n$ be the singular values of A . Clearly,

$$\begin{aligned}\|A\|_*^2 &= (\sigma_1 + \dots + \sigma_n)^2 \leq n(\sigma_1^2 + \dots + \sigma_n^2) = n(\text{tr}(AA^*)) \\ &= \sum_{i,j \in [n]} |a_{i,j}|^2 \leq n^2 - \sum_{i \in [r]} |N_i|^2 \leq n^2 - \frac{1}{r}n^2,\end{aligned}$$

completing the proof of (3). If equality holds in (3), then

$$(\sigma_1 + \dots + \sigma_n)^2 = n(\sigma_1^2 + \dots + \sigma_n^2) = (1 - 1/r)n^2,$$

and so $\sigma_1 = \dots = \sigma_n = \sqrt{(1 - 1/r)n}$, completing the proof of Theorem 2. \square

Remark 3 A matrix $A = [a_{i,j}]$ that makes (3) an equality has a long list of further properties, e.g.: r divides n ; the partition sets are of size n/r ; if an entry $a_{i,j}$ is not in a diagonal block, then $|a_{i,j}| = 1$; and most importantly, $AA^* = (1 - 1/r)nI_n$. Thus, the rows of A are orthogonal, and so are its columns. It seems hard to find for which r and n such matrices exist.

Next, from Theorem 2 we deduce a similar bound for nonnegative matrices, in particular, for graphs.

Theorem 4 Let $n \geq r \geq 2$, and let A be an $n \times n$ nonnegative matrix with $\|A\|_{\max} \leq 1$. If A is r -partite, then

$$\|A\|_* \leq \frac{n^{3/2}}{2} \sqrt{1 - 1/r} + (1 - 1/r)n.$$

Proof For each $i \in [r]$, set $n_i = |N_i|$, and write K for the matrix obtained from J_n by zeroing $J[N_i, N_i]$ for all $i \in [r]$. Note that K is the adjacency matrix of the complete r -partite graph with vertex classes N_1, \dots, N_r . Since K has no positive eigenvalue other than the largest one $\lambda_1(K)$, we see that $\|K\|_* = 2\lambda_1(K)$. A result of Cvetković [2] implies that $\lambda_1(K) \leq (1 - 1/r)n$, and so $\|K\|_* \leq 2(1 - 1/r)n$.

Now, let $B := 2A - K$, and note that the matrix B and the sets N_1, \dots, N_r satisfy the premises of Theorem 2; hence, using the triangle inequality, we find that

$$n^{3/2} \sqrt{1 - 1/r} \geq \|B\|_* \geq \|2A - K\|_* \geq 2\|A\|_* - \|K\|_* \geq 2\|A\|_* - 2(1 - 1/r)n,$$

completing the proof of Theorem 4. \square

Note that the matrix A in Theorems 2 and 4 needs not be symmetric; nonetheless, the following immediate corollary gives precisely Koolen and Moulton's bound (2) if $r = 2$.

Corollary 5 Let $n \geq r \geq 2$. If G is an r -partite graph of order n , then

$$\|G\|_* \leq \frac{n^{3/2}}{2} \sqrt{1 - 1/r} + (1 - 1/r)n. \quad (4)$$

2.1 Upper bounds for graphs

For $r \geq 3$ bound (4) can be somewhat improved by more involved methods. To this end, first we shall give an upper bound on the trace norm of an r -partite graph with n vertices and m edges. Hereafter, $\lambda_i(G)$ stands for the i 'th largest eigenvalue of the adjacency matrix of a graph G .

Theorem 6 *Let $n > r > 2$ and $2m \geq r^2 n$. If G is an r -partite graph with n vertices and m edges, then*

$$\|G\|_* \leq \frac{4m}{n} + \sqrt{(n-r) \left(2m - \frac{r}{r-1} \left(\frac{2m}{n} \right)^2 \right)}. \quad (5)$$

Equality holds if and only if the following three conditions are met:

- (i) G is a regular graph;
- (ii) the $r-1$ smallest eigenvalues of G satisfy

$$\lambda_n(G) = \dots = \lambda_{n-r+2}(G) = -\frac{2m}{(r-1)n};$$

- (iii) the eigenvalues $\lambda_2(G), \dots, \lambda_{n-r+1}(G)$ satisfy

$$\lambda_2^2(G) = \dots = \lambda_{n-r+1}^2(G) = \frac{1}{n-r} \left(2m - \frac{r}{r-1} \left(\frac{2m}{n} \right)^2 \right).$$

Proof Let the graph G satisfy the premises of the theorem, and for short, write λ_i for $\lambda_i(G)$. Using the fact that $\lambda_1^2 + \dots + \lambda_n^2 = 2m$ and the AM-QM inequality, we see that

$$\begin{aligned} \|G\|_* &= \lambda_1 + \sum_{i=2}^{n-r+1} |\lambda_i| + \sum_{i=n-r+2}^n |\lambda_i| \leq \lambda_1 + \sum_{i=n-r+2}^n |\lambda_i| + \sqrt{(n-r) \sum_{i=2}^{n-r+1} \lambda_i^2} \\ &= \lambda_1 + \sum_{i=n-r+2}^n |\lambda_i| + \sqrt{(n-r) \left(2m - \lambda_1^2 - \sum_{i=n-r+2}^n |\lambda_i|^2 \right)} \\ &\leq \lambda_1 + \sum_{i=n-r+2}^n |\lambda_i| + \sqrt{(n-r) \left(2m - \lambda_1^2 - \frac{1}{r-1} \left(\sum_{i=n-r+2}^n |\lambda_i| \right)^2 \right)}. \end{aligned}$$

Since G is r -partite, Hoffman's bound [7] implies that

$$\lambda_1 \leq |\lambda_{n-r+2}| + \dots + |\lambda_n|.$$

Now, letting $x = \lambda_1$, $y = |\lambda_{n-r+2}| + \dots + |\lambda_n|$, and $2m = A$, we maximize the function

$$f(x, y) := x + y + \sqrt{(n-r) \left(A - x^2 - \frac{1}{r-1} y^2 \right)},$$

subject to the constraints

$$n \geq r \geq 3, \quad A \geq r^2 n, \quad y \geq x \geq A/n, \quad x^2 + \frac{1}{r-1} y^2 \leq A. \quad (6)$$

We shall show that $f(x, y) < f(A/n, A/n)$, unless $y = x = A/n$. To this end, first we show that $f(x, y)$ is decreasing in y if $y > x$. Assume the opposite, that is to say, there are x and y , satisfying (6), with $x < y$ and

$$\frac{\partial f(x, y)}{\partial y} = 1 - \frac{(n-r)y/(r-1)}{\sqrt{(n-r)(A - x^2 - \frac{1}{r-1}y^2)}} \geq 0.$$

After some algebra we obtain

$$\begin{aligned} A &\geq \frac{1}{r-1} y^2 + (n-r) \frac{y^2}{(r-1)^2} + x^2 > \left(\frac{n-1}{(r-1)^2} + 1 \right) x^2 > \left(\frac{n-1}{(r-1)^2} + 1 \right) \frac{A^2}{n^2} \\ &\geq \left(\frac{n-1}{(r-1)^2} + 1 \right) r^2 \frac{A}{n} > A, \end{aligned}$$

a contradiction, proving that $f(x, y) < f(x, x)$, unless $y = x$.

Next, we maximize the function

$$g(x) := 2x + \sqrt{(n-r) \left(A - \frac{r}{r-1} x^2 \right)},$$

subject to the constraints

$$n > r \geq 3, \quad A \geq r^2 n, \quad x \geq A/n, \quad \frac{r}{r-1} x^2 \leq A. \quad (7)$$

We shall show that $g(x) < g(A/n)$, unless $x = A/n$. To this end, we shall prove that $g(x)$ is decreasing in x , whenever $x > A/n$. Assume the opposite, that is to say, there is an x , satisfying (7), with $x < A/n$ and

$$\frac{dg(x)}{dx} = 2 - \frac{(n-r)rx/(r-1)}{\sqrt{(n-r)(A - \frac{r}{r-1}x^2)}} \geq 0.$$

After some algebra we get

$$\begin{aligned} 4A &\geq \left(\frac{(n-r)r^2}{(r-1)^2} + \frac{r}{r-1} \right) x^2 > \left(\frac{(n-r)r^2}{(r-1)^2} + \frac{r}{r-1} \right) \frac{A^2}{n^2} \geq \left(\frac{(n-r)r^2}{(r-1)^2} + \frac{r}{r-1} \right) r^2 \frac{A}{n} \\ &= \left(r - \frac{r^2 - r + 1}{n} \right) \frac{r^3}{(r-1)^2} A > \left(r - \frac{r^2 - r + 1}{r} \right) \frac{r^3}{(r-1)^2} A = \frac{r^2}{r-1} A > 4A. \end{aligned}$$

This contradiction implies that $g(x) < g(A/n)$, unless $x = A/n$. Therefore, $f(x, y) < f(A/n, A/n)$ unless $y = x = A/n$. This inequality implies (5). It also implies that if equality holds in (5) then $\lambda_1 = 2m/n$, and so clause (i) follows. Further, equality in (5) implies that

$$\lambda_{n-r+2} + \cdots + \lambda_n = -\frac{2m}{n} \quad \text{and} \quad \sum_{i=n-r+2}^n \lambda_i^2 = \frac{1}{r-1} \left(\sum_{i=n-r+2}^n |\lambda_i| \right)^2$$

and so clause (ii) follows as well. Finally, equality in (5) implies clause (iii) in view of

$$\sum_{i=2}^{n-r+1} \lambda_i^2 = \frac{1}{n-r} \left(\sum_{i=2}^{n-r+1} |\lambda_i| \right)^2 \quad \text{and} \quad \sum_{i=2}^{n-r+1} \lambda_i^2 = 2m - \lambda_1^2 - \sum_{i=n-r+2}^n \lambda_i^2,$$

completing the proof of Theorem 6. \square

Next, we maximize bound (5) over m and get a bound that depends only on r and n .

Theorem 7 *Let $r \geq 2$ and $n \geq 4(r-1)^2$. If G is an r -partite graph of order n , then*

$$\|G\|_* \leq \frac{n(n-r)}{2\sqrt{(n-r)\frac{r}{r-1}+4}} + \frac{(r-1)n}{r} + \frac{2(r-1)n}{r\sqrt{(n-r)\frac{r}{r-1}+4}}. \quad (8)$$

Equality holds if and only if the following three conditions are met:

(i) G is a regular graph of degree

$$\left(1 + \frac{2}{\sqrt{(n-r)\frac{r}{r-1}+4}} \right) \frac{(r-1)n}{2r};$$

(ii) the $r-1$ smallest eigenvalues of G satisfy

$$\lambda_n(G) = \cdots = \lambda_{n-r+2}(G) = - \left(1 + \frac{2}{\sqrt{(n-r)\frac{r}{r-1}+4}} \right) \frac{n}{2r}.$$

(iii) the eigenvalues $\lambda_2(G), \dots, \lambda_{n-r+1}(G)$ satisfy

$$|\lambda_2(G)| = \cdots = |\lambda_{n-r+1}(G)| = \frac{n}{2\sqrt{(n-r)\frac{r}{r-1}+4}}.$$

Proof If $2m \geq r^2n$, we maximize the function

$$f(x) := 2x + \sqrt{(n-r) \left(xn - \frac{r}{r-1}x^2 \right)},$$

subject to

$$(r-1)^2/2 \leq x \leq (1-1/r)n,$$

and find that $f(x)$ attains a maximum for

$$x = \left(1 + \frac{2}{\sqrt{(n-r)\frac{r}{r-1} + 4}}\right) \frac{(r-1)n}{2r},$$

which gives precisely (8).

If $2m < r^2n$, we use the crude estimate by the AM-QM inequality,

$$\|G\|_* \leq \sum_{i=1}^n |\lambda_i| < \sqrt{n \sum_{i=1}^n \lambda_i^2} = \sqrt{2mn} < rn.$$

Now (8) follows by

$$r < \frac{(n-r)}{2\sqrt{(n-r)\frac{r}{r-1} + 4}} + \frac{r-1}{r},$$

which is equivalent to

$$\sqrt{\frac{r-1}{r-1/4} + \frac{(r-1)^2}{r^2(r-1/4)^2}} < \frac{r(r-1)}{r^2-r+1} \quad (9)$$

For the left side of (9) we get

$$\begin{aligned} \sqrt{\frac{r-1}{r-1/4} + \frac{(r-1)^2}{r^2(r-1/4)^2}} &< \sqrt{1 - \frac{3}{4r-1} + \frac{1}{r^2}} < 1 - \frac{3}{2(4r-1)} + \frac{1}{2r^2} \\ &< 1 - \frac{3}{8r} + \frac{1}{2r^2}. \end{aligned}$$

For the right side of (9) we see that

$$\frac{r(r-1)}{r^2-r+1} = 1 - \frac{1}{r^2-r+1} > 1 - \frac{1}{r^2-r}.$$

Now, (9) follows from

$$1 - \frac{1}{r^2-r} > 1 - \frac{3}{8r} + \frac{1}{2r^2},$$

which is true for $r \geq 3$.

Clauses (i), (ii), and (iii) are just a restatement of last part of Theorem 6, so we omit them.

□

Remark 8 *It is possible that bound (8) is exact for infinitely many r and n . In general, it can be shown, that (8) is better than (4) as long as $n > 4(r-1)^2$, but the difference between their right sides never exceeds some constant that is independent of n . That is to say, (4) is exact within a linear term in n .*

3 Constructions

Recall that an *Hadamard matrix* of order n is an $n \times n$ matrix H with entries of modulus 1 and such that $HH^* = nI_n$; hence, all singular values of H are equal to \sqrt{n} . Also, a *conference matrix* of order n is an $n \times n$ matrix C with zero diagonal, with off-diagonal entries of modulus 1, and such that $CC^* = (n-1)I_n$; hence all singular values of C are equal to $\sqrt{n-1}$. For details on Hadamard and conference matrices the reader is referred to [3, 8]. We shall write \otimes for the Kronecker (tensor) multiplication of matrices.

First, we show that bound (3) in Theorem 2 is best possible for infinitely many n , whenever r is the order of a conference matrix.

Theorem 9 *Let r be the order of a conference matrix of order r , and let k be the order of an Hadamard matrix. There exists an r -partite matrix A of order $n = rk$ with $\|A\|_{\max} = 1$ and*

$$\|A\|_* = n^{3/2} \sqrt{1 - 1/r}.$$

Proof Let C be a conference matrix of order r and H be an Hadamard matrix of order k . Let $A := C \otimes H$, and partition $[rk]$ into r consecutive segments N_1, \dots, N_r of length k ; we see that $\|A\|_{\max} = 1$ and that $A[N_i, N_i] = 0$ for any $i \in [r]$. Finally, we find that

$$\|A\|_* = \|C \otimes H\|_* = \|C\|_* \|H\|_* = r\sqrt{r-1}k^{3/2} = n^{3/2} \sqrt{1 - 1/r},$$

completing the proof of Theorem 9. □

Next, a modification of the above construction provides some matching lower bounds for Theorems 4 and 7, and Corollary 5.

Theorem 10 *Let r be the order of a real symmetric conference matrix. If k is the order of a real symmetric Hadamard matrix, then there is an r -partite graph G of order $n = rk$ with*

$$\|G\|_* \geq \frac{n^{3/2}}{2} \sqrt{1 - 1/r} - (1 - 1/r)n.$$

Proof Let C be a real symmetric conference matrix of order r , and let H be a real symmetric Hadamard matrix of order k . Let $B := C \otimes H$, and partition $[rk]$ into r consecutive segments N_1, \dots, N_r of length k . We see that $B[N_i, N_i] = 0$ for any $i \in [r]$, and also $B[N_i, N_j]$ is a $(-1, 1)$ -matrix whenever $i, j \in [r]$ and $i \neq j$. Finally, let

$$A := \frac{1}{2} (B + K_r \otimes J_k),$$

and note that A is a symmetric $(0, 1)$ -matrix, and $A[N_i, N_i] = 0$ for any $i \in [r]$. Hence A is the adjacency matrix of an r -partite graph G of order n . Note that the singular values of B are equal to $\sqrt{k(r-1)} = \sqrt{(1-1/r)n}$. Thus, using the triangle inequality, we find that

$$\|(B + K_n \otimes J_k)\|_* \geq \|B\|_* - \|K_r \otimes J_k\|_* \geq n^{3/2} \sqrt{1 - 1/r} - 2(r-1)k,$$

and so,

$$\|G\|_* \geq \frac{n^{3/2}}{2} \sqrt{1 - 1/r} - (1 - 1/r) n,$$

completing the proof of Theorem 10. \square

Remark 11 *Complex Hadamard matrices of order n exists for any n . This is not true for real Hadamard matrices, although there are various constructions of such matrices, e.g., Paley's constructions:*

If q is an odd prime power, then there is a real conference matrix of order $q + 1$, which is symmetric if $q \equiv 1 \pmod{4}$; there is a real Hadamard matrix of order $q + 1$ if $q \equiv 3 \pmod{4}$; there is a real symmetric Hadamard matrix of order $2(q + 1)$ if $q \equiv 1 \pmod{4}$.

4 Asymptotics

Write \mathcal{C}_k for the class of all complex k -partite matrices A with $\|A\|_{\max} \leq 1$, and let $\mathcal{H}_k \subset \mathcal{C}_k$ be the subclass of the Hermitian matrices in \mathcal{C}_k . Likewise, write \mathcal{R}_k for the class of the real k -partite matrices A with $\|A\|_{\max} \leq 1$, and let $\mathcal{S}_k \subset \mathcal{R}_k$ be the subclass of the symmetric elements of \mathcal{R}_k .

Further, write $n(A)$ for the order of a square matrix A , and for any class of square matrices \mathcal{X} , let $\mathcal{X}(n)$ stand for the subclass of the elements of \mathcal{X} with $n(A) = n$.

With this notation let us define the functions $c_k(n)$, $h_k(n)$, $r_k(n)$, and $s_k(n)$ as

$$\begin{aligned} c_k(n) &:= \max \{ \|A\|_* : A \in \mathcal{C}_k(n) \}, & h_k(n) &:= \max \{ \|A\|_* : A \in \mathcal{H}_k(n) \}, \\ r_k(n) &:= \max \{ \|A\|_* : A \in \mathcal{R}_k(n) \}, & s_k(n) &:= \max \{ \|A\|_* : A \in \mathcal{S}_k(n) \}. \end{aligned}$$

Theorem 9 shows that if there is a complex conference matrix of order k , then $c_k(n) = n^{3/2} \sqrt{1 - 1/r}$; similar statements hold also for $h_k(n)$, $r_k(n)$, and $s_k(n)$. However, conference matrices are rare and it is difficult to determine $c_k(n)$, $h_k(n)$, $r_k(n)$, and $s_k(n)$ for any k . Thus, in what follows, we shall prove the possibility for certain asymptotics in n for each of these functions.

For a start, Theorem 2 implies that if $A \in \mathcal{C}_k$, then $\|A\|_* \leq (n(A))^{3/2}$. Therefore, for any $k \geq 2$, it is possible to define the constants c_k , h_k , r_k , and s_k as

$$\begin{aligned} c_k &:= \sup \left\{ \frac{\|A\|_*}{(n(A))^{3/2}} : A \in \mathcal{C}_r \right\}, & h_k &:= \sup \left\{ \frac{\|A\|_*}{(n(A))^{3/2}} : A \in \mathcal{H}_r \right\}, \\ r_k &:= \sup \left\{ \frac{\|A\|_*}{(n(A))^{3/2}} : A \in \mathcal{R}_r \right\}, & s_k &:= \sup \left\{ \frac{\|A\|_*}{(n(A))^{3/2}} : A \in \mathcal{S}_r \right\}. \end{aligned}$$

Note again that Theorem 9 yields $c_k = \sqrt{1 - 1/r}$ if there is a complex conference matrix of order k , and similar statements can be proved also for h_k , r_k , and s_k . However, the main use

of c_k , h_k , r_k , and s_k is to provide some asymptotics for $c_k(n)$, $h_k(n)$, $r_k(n)$, and $s_k(n)$: indeed the above definitions imply that

$$c_k(n) \leq c_k n^{3/2}, \quad h_k(n) \leq h_k n^{3/2}, \quad r_k(n) \leq r_k n^{3/2}, \quad \text{and} \quad s_k(n) \leq s_k n^{3/2},$$

and, as it turns out, these inequalities are tight.

Theorem 12 *For any $k \geq 2$, the functions $c_k(n)$, $h_k(n)$, $r_k(n)$, and $s_k(n)$ satisfy:*

$$\lim_{n \rightarrow \infty} \frac{c_k(n)}{n^{3/2}} = c_k, \quad \lim_{n \rightarrow \infty} \frac{h_k(n)}{n^{3/2}} = h_k, \quad \lim_{n \rightarrow \infty} \frac{r_k(n)}{n^{3/2}} = r_k, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{s_k(n)}{n^{3/2}} = s_k.$$

Proof We shall prove only the statement for $h_k(n)$, as the proofs of the other cases are essentially the same. Let $\varepsilon > 0$ and choose a matrix $A \in \mathcal{H}_k$, say with $n(A) = n$, such that

$$\frac{\|A\|_*}{n^{3/2}} > h_k - \frac{\varepsilon}{2}.$$

We shall show that for m sufficiently large

$$\frac{h_k(m)}{m^{3/2}} > h_k - \varepsilon. \tag{10}$$

To this end, let q be any prime with $q \equiv 1 \pmod{4}$, and recall that Paley's construction yields a real symmetric Hadamard matrix H of order $2(q+1)$. Set $B := A \otimes H$ and note that B is a Hermitian k -partite matrix with $\|B\|_{\max} \leq 1$. Hence $B \in \mathcal{H}_k(2(q+1)n)$. Thus, for any integer $m = 2(q+1)n$, where q is a prime with $q \equiv 1 \pmod{4}$, we see that

$$\frac{h_k(m)}{m^{3/2}} \geq \frac{\|B\|_*}{(n(B))^{3/2}} = \frac{\|A \otimes H\|_*}{n^{3/2} (n(H))^{3/2}} = \frac{\|A\|_*}{n^{3/2}} > h_k - \frac{\varepsilon}{2}. \tag{11}$$

Now, let m be any integer and let q be the largest prime with $q \equiv 1 \pmod{4}$ such that $2(q+1)n < m$. Set $2(q+1)n := t$ and let $C \in \mathcal{H}_k(t)$ be a matrix with $\|C\|_* = h_k(t)$. Hence (11) implies that

$$\frac{\|C\|_*}{n^{3/2} (2(q+1))^{3/2}} > h_k - \frac{\varepsilon}{2}.$$

Let $[t] = N_1 \cup \dots \cup N_k$ be the partition of the index set of C such that $C[N_i, N_i] = 0$ for any $i \in [k]$. Define an $m \times m$ matrix B , by extending C with $m - t$ zero columns and rows and letting $N'_k := N_k \cup ([m] \setminus [t])$. Thus, $B[N'_k, N'_k] = 0$, and so B is k -partite. Clearly $\|B\|_{\max} \leq 1$, and therefore $B \in \mathcal{H}_k(m)$. Further, we find that

$$\|B\|_* \geq \|C\|_* \geq \left(h_k - \frac{\varepsilon}{2}\right) n^{3/2} (2(q+1))^{3/2},$$

and hence,

$$\frac{\|B\|_*}{m^{3/2}} > \left(h_k - \frac{\varepsilon}{2}\right) \frac{n^{3/2} (2(q+1))^{3/2}}{m^{3/2}}.$$

A result about the distribution of primes [1] implies that if q is large enough, then there is a prime p with $p \equiv 1 \pmod{4}$ such that $q < p < q + q^{11/20}$. Hence

$$m < 2(q + q^{11/20} + 1)n,$$

and so, using Bernoulli's inequality,

$$\begin{aligned} \frac{\|B\|_*}{m^{3/2}} &> \left(h_k - \frac{\varepsilon}{2}\right) \frac{(q+1)^{3/2}}{(q + q^{11/20} + 1)^{3/2}} > \left(h_k - \frac{\varepsilon}{2}\right) \left(1 - \frac{q^{11/20}}{q+1}\right) \\ &> \left(h_k - \frac{\varepsilon}{2}\right) \left(1 - \frac{3}{2}q^{-9/20}\right). \end{aligned}$$

Therefore, if m is large enough, we see that (10) holds. Since $h_k(m) \leq h_k m^{3/2}$, it follows that

$$\lim_{n \rightarrow \infty} \frac{h_k(n)}{n^{3/2}} = h_k,$$

completing the proof of the theorem. □

Particularly challenging is the following problem:

Problem 13 Find c_3 .

4.1 Two general bounds on $\|G\|_*$

Since the set of known conference and Hadamard matrices is quite sparse, the following two explicit estimates may be useful.

Proposition 14 *For any $r > 2$, there is an $n_0(r)$, such that if $n > n_0(r)$, then there is an r -partite graph G of order n with*

$$\|G\|_* > \frac{n^{3/2}}{2} (1 - 1/\sqrt{r}) - n^{21/20}.$$

For large r this bound can be improved using results on prime distribution:

Proposition 15 *For any sufficiently large r , there is an $n_0(r)$, such that if $n > n_0(r)$, then there is a graph G of order n , such that*

$$\|G\|_* > \frac{n^{3/2}}{2} \sqrt{1 - 1/(r - r^{11/20})}.$$

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